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Pair Mean Cordial Labeling of Total Graph of Some Graphs and Prism Related Graphs

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Abstract

Let $G = (V, E)$ be a graph with p vertices and q edges. Define $\rho = \begin{cases} \frac{p}{2} & p \text{ is even} \\ \frac{p-1}{2} & p \text{ is odd,} \end{cases}$ and $M = \{\pm 1, \pm 2, \cdots \pm \rho\}$. Consider a mapping $\lambda : V \to M$ by assigning different labels in M to the different elements of V when p is even and different labels in M to $p-1$ elements of V and repeating a label for the remaining one vertex when p is odd. The labeling as defined above is said to be a pair mean cordial labeling if for each edge uv of G, there is a labeling $\frac{\lambda(u)+\lambda(v)}{2}$ if $\lambda(u) + \lambda(v)$ is even and $\frac{\lambda(u)+\lambda(v)+1}{2}$ if $\lambda(u) + \lambda(v)$ is odd such that $|\bar{S}_{\lambda_1} - \bar{S}_{\lambda_1^c}| \leqslant 1$ where $\bar{S}_{\lambda_1}$ and $\bar{S}_{\lambda_1^c}$ respectively denote the number of edges labeled with 1 and the number of edges not labeled with 1. A graph G for which there is a pair mean cordial labeling is called a pair mean cordial graph (PMC-graph). In this paper, we investigate the pair mean cordial labeling behavior of some graphs like total graph of path, cycle, star, crown and comb and Also we examine the pair mean cordial labeling behavior of triangular winged prism graph, rectangular winged prism graph, $W-$ graph and irregular pentagonal snake.

Keywords: path, cycle, star, crown, comb, total graph.

2020 MSC: 05C78.

1. Introduction

All graphs in this study are the finite, simple and undirected. The majority of citations in the literature credit Rosa's work [17] from 1967 as the starting point for graph labeling and of graph labelling related graphs as in [1, 2, 3, 5, 8, 9, 16, 18, 19, 20]. We follow the basic notation and terminology of graph theory as in Harary [7]. Cahit defines cordial labeling, a variation of both graceful and harmonious labeling in [4], and For extensive survey on graph labeling we refer to Gallian [6]. A study on pair mean cordial labeling of some special graphs is referred to [10, 11, 12, 13, 14, 15]. In this present paper, we investigate the pair mean cordial labeling behavior of some graphs like total graph of path, cycle, star, crown and comb and also we examine the pair mean cordial labeling behavior of triangular winged prism graph, rectangular winged prism graph, $W-$ graph and irregular pentagonal snake.

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2. Preliminaries

Definition 2.1. [10] The comb graph $P_n \odot K_1$ is a graph by connecting each vertex in the path P_n with a pendant edge. There are $2n$ vertices and $2n-1$ edges in the comb graph.

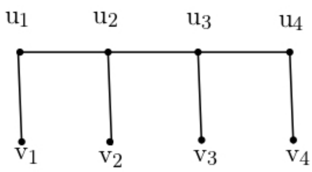

Figure 1: The comb graph $P_4 \odot K_1$.

Definition 2.2. [12] A graph obtained by adding a single pendent edge to each vertex of a cycle C_n is called a crown graph and it is denoted by $C_n \odot K_1$.

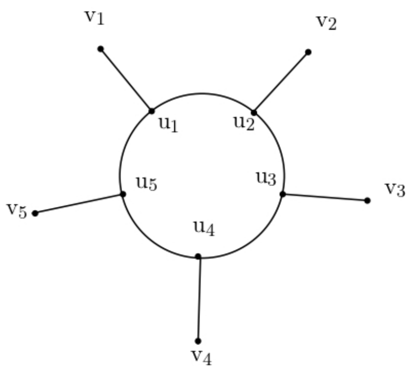

Figure 2: The crown graph $C_5 \odot K_1$.

Definition 2.3. [6] The total graph of a graph G, denoted by $T(G)$ is define as, the vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $T(G)$ are adjacent in $T(G)$ in case one of the following condition holds: (i) x, y are in $V(G)$ and x is adjacent to y in G. (ii) x, y are in $E(G)$ and x, y is adjacent in G. (iii) x is in $V(G)$, y is in $E(G)$ and x, y are incident in G.

Definition 2.4. [10] A star graph S_n is a complete bipartite graph $K_{1,n}$ with one internal node and n – leaves. A star having three edges is called as claw.

Definition 2.5. [10] The bistar graph $B_{m,n}$ is a graph derived by joining the centre node by an edge of two-star graph $K_{1,n}$ and $K_{1,m}$.

Definition 2.6. [9] The corona graph $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and n copies of G_2 and joining i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2, where G_1 is graph of order n.

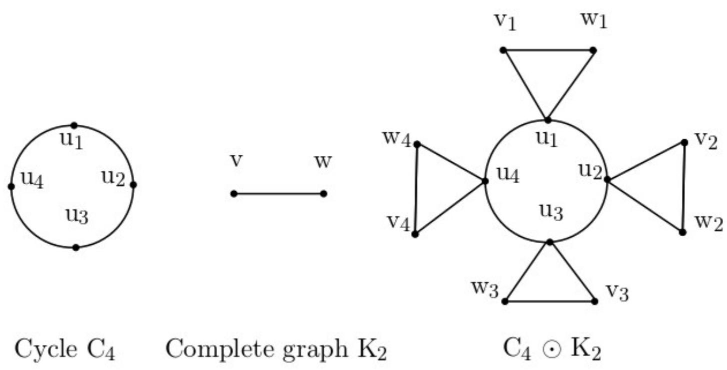

Figure 3: $C_4 \odot K_2$.

Definition 2.7. [16] Let P_n be a path with consecutive vertices $u_1u_2\dots u_n$. An irregular pentagonal snake graph IP_n is obtained from the path P_n and new vertices x_i, y_i, v_i, $1 \leqslant i \leqslant n-2$ and edges $x_iv_i, y_iv_i, u_ix_i, y_iu_{i+2}$, $1 \leqslant i \leqslant n-2$.

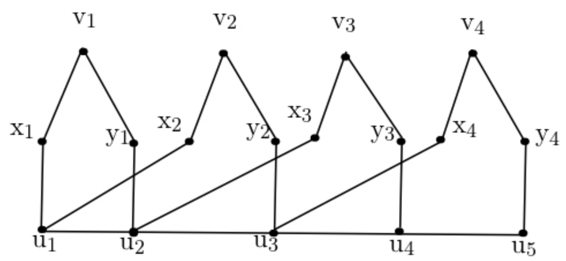

Figure 4: The irregular pentagonal snake graph IP_n

Definition 2.8. [6] A $W-$graph WB_n is a graph obtained from the bistar $B_{n,n}$ by subdividing the central edge.

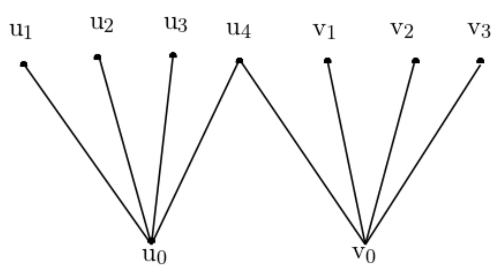

Figure 5: The $W-$graph WB_4 .

Definition 2.9. [20] The triangular winged prism graph, denoted by $TWPR_n$ is a graph with vertex set $V(TWPR_n) = \{u_i, v_i, w_i \mid 1 \leqslant i \leqslant n\}$ and edge set $E(TWPR_n) = \{u_iv_i, v_iw_i \mid 1 \leqslant i \leqslant n\} \cup \{u_iu_{i+1}, u_nu_1, v_iv_{i+1}, v_nv_1$

$1 \leqslant i \leqslant n-1\}$.

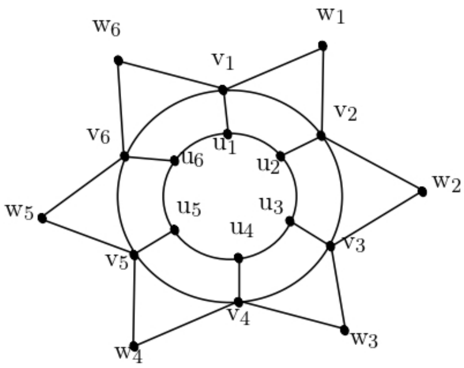

Figure 6: The triangular winged prism graph $TWPR_6$

Definition 2.10. [20] The rectangular winged prism graph, denoted by $RWPR_n$ is a graph with vertex set $V(RWPR_n) = \{u_i, v_i, x_i, y_i \mid 1 \leqslant i \leqslant n\}$ and edge set $E(RWPR_n) = \{u_iv_i, v_ix_i, x_iy_i \mid 1 \leqslant i \leqslant n\} \cup \{u_iu_{i+1}, u_nu_1, v_iv_{i+1}, v_nv_1, y_iv_{i+1}, y_nv_1 \mid 1 \leqslant i \leqslant n-1\}$.

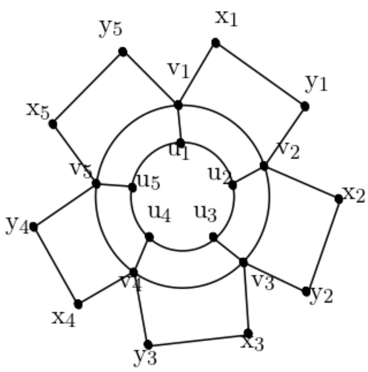

Figure 7: The rectangular winged prism graph $RWPR_5$.

3. Main Results

Theorem 3.1. The total graph of the path P_n, $T(P_n)$ is a PMC-graph only for all values of n except for $n = 3$.

Proof. Let $V(T(P_n)) = \{u_i, v_j \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n-1\}$ and $E(T(P_n)) = \{u_iu_{i+1}, u_iv_i, u_{i+1}v_i \mid 1 \leqslant i \leqslant n-1\} \cup \{v_iv_{i+1} \mid 1 \leqslant i \leqslant n-2\}$ respectively be the vertex set and edge set of the total graph of the path P_n, $T(P_n)$. Hence, $T(P_n)$ has $4n-5$ edges and $2n-1$ vertices. We consider three cases.
Case (i) : For $n = 1, 2$
Note that $T(P_1) \simeq P_1$, the path P_1 is PMC-graph[10]. Then, $T(P_2) \simeq C_3$, the cycle C_3 is PMC-graph[10].
Case (ii) : For $n = 3$

Suppose, $T(P_3)$ is a PMC-graph. If the edge uv get the label 1, the only possibilities are $\lambda(u)+\lambda(v)=1$ or $\lambda(u)+\lambda(v)=2$. Hence, the maximum number of edges labeled with 1 is 2. That's $\bar{S}_{\lambda_1} \leqslant 2$. More over, $\bar{S}_{\lambda_1^c} \geqslant 5$. Therefore, $\bar{S}_{\lambda_1^c} - \bar{S}_{\lambda_1} \geqslant 5-2=3>1$, a contradiction.
Case (iii) : For $n \geqslant 4$
Define $\lambda(u_n) = 1$ and $\lambda(v_{n-1}) = -n+1$. Assign the labels $2, 3, \ldots, n-1$ according to the vertices $u_1, u_2, \ldots, u_{n-2}$ and fix the label $-n+2$ with u_{n-1}. Now, give the labels $-1, -2, \ldots, -n+3$ according to the vertices $v_1, v_2, \ldots, v_{n-3}$ and fix the label $n-1$ with v_{n-2}. Eventually, $\bar{S}_{\lambda_1} = 2n-3$ and $\bar{S}_{\lambda_1^c} = 2n-2$. □

Example 3.2. Figure 8 illustrates the graph $T(P_4)$ is a PMC-graph.

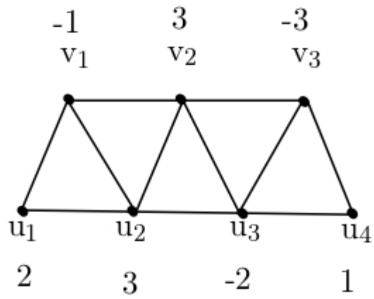

Figure 8: PMC-labeling of the total graph $T(P_4)$.

Theorem 3.3. The total graph of the cycle C_n, $T(C_n)$ is not PMC-graph for all values of $n \geqslant 3$

Proof. Consider the total graph of the cycle C_n, $T(C_n)$. Denote $V(T(C_n)) = \{u_i, v_i \mid 1 \leqslant i \leqslant n\}$ and $E(T(C_n)) = \{u_iv_i \mid 1 \leqslant i \leqslant n\} \cup \{u_iu_{i+1}, v_iv_{i+1}, v_iu_{i+1}, u_nu_1, v_nv_1, v_nu_1 \mid 1 \leqslant i \leqslant n-1\}$ respectively by the vertex set and edge set of the total graph of the cycle, $T(C_n)$. Therefore, $T(C_n)$ has $4n$ edges and $2n$ vertices. If possible, let $T(C_n)$ be a PMC-graph. So, if the edge uv get the label 1, the only possibilities are $\lambda(u)+\lambda(v)=1$ or $\lambda(u)+\lambda(v)=2$. Then, the maximum number of edges labeled with 1 is $2n-3$. That's $\bar{S}_{\lambda_1} \leqslant 2n-3$. Thus, $\bar{S}_{\lambda_1^c} \geqslant 4n-(2n-3) = 2n+3$. Subsequently, $\bar{S}_{\lambda_1^c} - \bar{S}_{\lambda_1} \geqslant 2n+3-(2n-3) = 6 > 1$, a contradiction. □

Example 3.4. Figure 9 illustrates the total graph of cycle $T(C_5)$.

Theorem 3.5. The total graph of the crown $C_n \odot K_1$, $T(C_n \odot K_1)$ is not PMC-graph only for all values of $n \geqslant 3$

Proof. Let $V(T(C_n \odot K_1))) = \{u_i, v_i, x_i, y_i \mid 1 \leqslant i \leqslant n\}$ and $E(T(C_n \odot K_1))) = \{u_iv_i, u_ix_i, u_iy_i, x_iy_i, v_ix_i \mid 1 \leqslant i \leqslant n\} \cup \{v_iu_{i+1}, v_iv_{i+1}, v_ix_{i+1}, v_nu_1, v_nv_1, v_nx_1 \mid 1 \leqslant i \leqslant n-1\}$ respectively be the vertex set and edge set of the total graph of the crown, $T(C_n \odot K_1))$. Hence $T(C_n \odot K_1))$ has $8n$ edges and $4n$ vertices. If possible, let $T(C_n \odot K_1))$ be a PMC-graph. Then, the maximum number of the edges labeled with 1 is $4n-3$. That's $\bar{S}_{\lambda_1} \leqslant 4n-3$. Thus, $\bar{S}_{\lambda_1^c} \geqslant 8n-(4n-3) = 4n+3$. Subsequently, $\bar{S}_{\lambda_1^c} - \bar{S}_{\lambda_1} \geqslant 4n+3-(4n-3) = 6 > 1$, a contradiction.

□

Example 3.6. Figure 10 illustrates the total graph of crown $T(C_3 \odot K_1)$.

Theorem 3.7. The total graph of the comb $P_n \odot K_1$, $T(P_n \odot K_1)$ is a PMC-graph only for all values of $n \leqslant 2$.

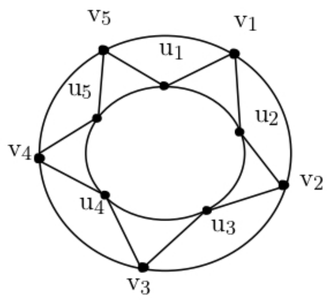

Figure 9: Total graph of cycle $T(C_5)$

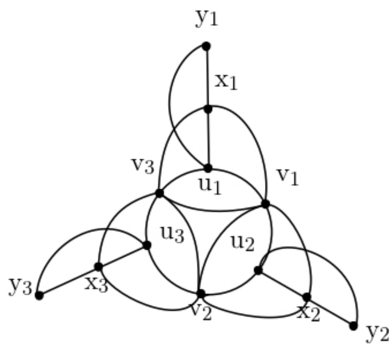

Figure 10: Total graph of crown $T(C_3 \odot K_1)$

Proof. Let $V(T(P_n \odot K_1) = \{u_i, v_i, x_i, y_j \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n-1\}$ and $E(T(P_n \odot K_1) = \{u_iv_i, u_ix_i, x_iv_i \mid 1 \leqslant i \leqslant n\} \cup \{x_iy_i, v_iy_i, v_iv_{i+1}, y_ix_{i+1}, y_iv_{i+1}, y_jy_{j+1} \mid 1 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant n-2\}$ respectively be the vertex set and edge set of the total graph of the comb, $T(P_n \odot K_1))$. Hence $T(P_n \odot K_1))$ has $8n$ edges and $4n$ vertices. We consider three cases:

Case (i) : For $n = 1$

Note that $T(P_n \odot K_1) \simeq C_3$, the cycle C_3 is PMC-graph [10].

Case (ii) : For $n = 2$

Consider the total graph of the comb, $T(P_2 \odot K_1)$. Figure 11 illustrates the graph $T(P_2 \odot K_1)$ is a PMC-graph.

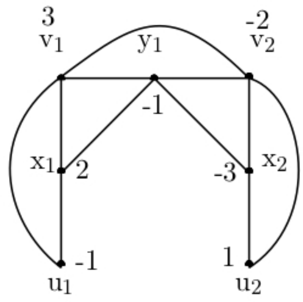

Figure 11: Total graph of comb $T(P_2 \odot K_1)$ is a PMC-graph.

Case (iii) : For $n \geqslant 3$
If possible, let $T(P_n \odot K_1))$ be a PMC-graph. Then, the maximum number of the edges labeled with 1 is $4n-3$. That's $\bar{S}_{\lambda_1} \leqslant 4n-3$. Also, $\bar{S}_{\lambda_1^c} \geqslant 9n-7-(4n-3) = 5n-4$. Subsequently, $\bar{S}_{\lambda_1^c} - \bar{S}_{\lambda_1} \geqslant 5n-4-(4n-3) = n-1 \geqslant 2 > 1$, a contradiction.

□

Theorem 3.8. The total graph of the star S_n, $T(S_n)$ is a PMC-graph only for $n = 1$.

Proof. Let $V(T(S_n)) = \{u, u_i, v_i \mid 1 \leqslant i \leqslant n\}$ and $E(T(S_n)) = \{uu_i, uv_i, v_iu_i \mid 1 \leqslant i \leqslant n\} \cup \{v_iv_j \mid 1 \leqslant i, j \leqslant n, i \neq j\}$ respectively be the vertex set and edge set of the total graph of the star S_n, $T(S_n)$. It has $\frac{n^2+5n}{2}$ edges and $2n+1$ vertices. We consider four cases.
Case (i) : For $n = 1$
Note that $T(S_1) \simeq C_3$, the cycle C_3 is PMC-graph [10].
Case (ii) : For $n = 2$
If possible, let $T(S_n)$ be a PMC-graph. So, the Maximum number of the edges labeled with 1 is 2. That's $\bar{S}_{\lambda_1} \leqslant 2$. Thus, $\bar{S}_{\lambda_1^c} \geqslant 5$. Subsequently, $\bar{S}_{\lambda_1^c} - \bar{S}_{\lambda_1} \geqslant 5-2 = 3 > 1$, a contradiction.
Case (iii) : For $n = 3, 4$
Then, the Maximum number of the edges labeled with 1 is $2n-1$. That's $\bar{S}_{\lambda_1} \leqslant 2n-1$. Thus, $\bar{S}_{\lambda_1^c} \geqslant q-(2n-1) = \frac{n^2+n+2}{2} \geqslant 2$. Subsequently, $\bar{S}_{\lambda_1^c} - \bar{S}_{\lambda_1} \geqslant \frac{n^2+n+2}{2} - (2n-1) = \frac{n^2-3n+4}{2} \geqslant 2 > 1$, a contradiction.
Case (iv) : For $n \geqslant 5$
Moreover, the Maximum number of the edges labeled with 1 is $2n-1$. That's $\bar{S}_{\lambda_1} \leqslant n+5$. Next, $\bar{S}_{\lambda_1^c} \geqslant q-(n+5) = \frac{n^2+3n-10}{2} \geqslant 2$. Subsequently, $\bar{S}_{\lambda_1^c} - \bar{S}_{\lambda_1} \geqslant \frac{n^2+3n-10}{2} - (n+5) = \frac{n^2+n-20}{2} \geqslant 5 > 1$, a contradiction. □

Example 3.9. Figure 12 illustrates the total graph of star $T(S_3)$.

Theorem 3.10. The irregular pentagonal snake IP_n is a PMC-graph only for all values of $n \geqslant 3$.

Proof. Consider the irregular pentagonal snake IP_n. Denoting by $V(IP_n) = \{u_i, v_j, x_j, y_j \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n-2\}$ and $E(IP_n) = \{x_iv_i, y_iv_iu_ju_{j+1}, u_ix_i, y_iu_{i+2} \mid 1 \leqslant i \leqslant n-2, 1 \leqslant j \leqslant n-1\}$ respectively, the vertex and edge sets of the irregular pentagonal snake IP_n. Eventually, IP_n has $4n-6$ vertices and $5n-9$ edges. Note that $IP_3 \simeq C_5$, the cycle C_5 is a PMC-graph [15]. We consider two cases:
Case (i) : For odd n
Here, assign the labels $-1, -4, \ldots, \frac{-3n+7}{2}$ and $4, 7, \ldots, \frac{3n-7}{2}$ to the corresponding vertices $v_1, v_3, \ldots, v_{n-2}$ and $v_2, v_4, \ldots, v_{n-3}$. Then, assign the labels $2, 5, \ldots, \frac{3n-5}{2}$ and $-2, -5, \ldots, \frac{-3n+11}{2}$ to the corresponding vertices $x_1, x_3, \ldots, x_{n-2}$ and $x_2, x_4, \ldots, x_{n-3}$. Assign the labels $3, 6, \ldots, \frac{3n-3}{2}$ and $-3, -6, \ldots, \frac{-3n+9}{2}$ to the

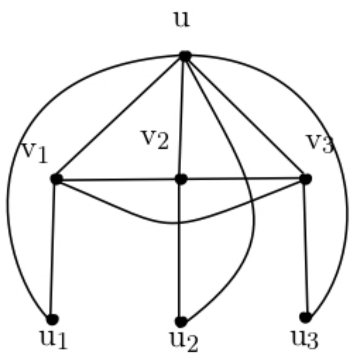

Figure 12: Total graph of star $T(S_3)$

corresponding vertices $y_1, y_3, \ldots, y_{n-2}$ and $y_2, y_4, \ldots, y_{n-3}$. Assign the labels $\frac{-3n+5}{2}, \frac{3n-1}{2}, \frac{-3n+3}{2}$ and $\frac{-3n+1}{2}, \frac{-3n-1}{2}, \ldots, -2n+3$ to the corresponding vertices u_1, u_2, u_3 and $u_4, u_6, \ldots, u_{n-1}$. Assign the labels $\frac{3n+1}{2}, \frac{3n+3}{2}, \ldots, 2n-3$ and 1 corresponding to the vertices $u_5, u_7, \ldots, u_{n-2}$ and u_n.
Case (ii) : For even n
Now, assign the labels $-1, -4, \ldots, \frac{-3n+10}{2}$ and $4, 7, \ldots, \frac{3n-4}{2}$ to the corresponding vertices $v_1, v_3, \ldots, v_{n-3}$ and $v_2, v_4, \ldots, v_{n-2}$. So, assign the labels $2, 5, \ldots, \frac{3n-8}{2}$ and $-2, -5, \ldots, \frac{-3n+8}{2}$ to the corresponding vertices $x_1, x_3, \ldots, x_{n-3}$ and $x_2, x_4, \ldots, x_{n-2}$. Further, assign the labels $3, 6, \ldots, \frac{3n-6}{2}$ and $-3, -6, \ldots, \frac{-3n+6}{2}$ to the corresponding vertices $y_1, y_3, \ldots, y_{n-3}$ and $y_2, y_4, \ldots, y_{n-2}$. Next, assign the labels $\frac{-3n+4}{2}, \frac{-3n+2}{2}, \ldots, -2n+3$ and $\frac{3n-2}{2}, \frac{3n}{2}, \ldots, 2n-3$ to the corresponding vertices $u_1, u_3, \ldots, u_{n-1}$ and $u_2, u_4, \ldots, u_{n-2}$. Fix the label 1 to u_n. □

Table 1: The following table demonstrates the vertex labelling λ is a PMC-labeling of the irregular pentagonal snake IP_n, for $n \geqslant 2$.

n	$\bar{S}_{\lambda_1}$	$\bar{S}_{\lambda_1^c}$
n is odd	$\frac{5n-9}{2}$	$\frac{5n-9}{2}$
n is even	$\frac{5n-10}{2}$	$\frac{5n-8}{2}$

Example 3.11. Figure 13 illustrates the PMC-labeling of the irregular pentagonal snake graph IP_5.

Theorem 3.12. The rectangular winged prism graph $RWPR_n$ is a PMC-graph only for all values of $n \geqslant 3$.

Proof. Consider the rectangular winged prism graph $RWPR_n$. Denoting by $V(RWPR_n) = \{u_i, v_i, x_i, y_i \mid 1 \leqslant i \leqslant n\}$ and $E(RWPR_n) = \{u_iv_i, v_ix_i, x_iy_i \mid 1 \leqslant i \leqslant n\} \cup \{u_iu_{i+1}, u_nu_1, v_iv_{i+1}, v_nv_1, y_iv_{i+1}, y_nv_1 \mid 1 \leqslant i \leqslant n-1\}$ respectively, the vertex and edge sets of the rectangular winged prism graph $RWPR_n$. Thus, $RWPR_n$ has $4n$ vertices and $6n$ edges. We consider two cases:
Case (i) : For odd n
Now, assign the labels $2, 5, \ldots, \frac{3n+1}{2}$ and $-2, -5, \ldots, \frac{-3n+5}{2}$ to the corresponding vertices $x_1, x_3, \ldots, x_n$ and $x_2, x_4, \ldots, x_{n-1}$. Then, assign the labels $-1, -4, \ldots, \frac{-3n+1}{2}$ and $4, 7, \ldots, \frac{3n-1}{2}$ to the corresponding vertices $y_1, y_3, \ldots, y_n$ and $y_2, y_4, \ldots, y_{n-1}$. Fix the label $\frac{3n+3}{2}$ with v_1. Assign the labels $3, 6, \ldots, \frac{3n-3}{2}$ and $-3, -6, \ldots, \frac{-3n+3}{2}$ to the corresponding vertices $v_2, v_4, \ldots, v_{n-1}$ and $v_3, v_5, \ldots, v_n$. Next, assign the labels $\frac{-3n-1}{2}, \frac{-3n-3}{2}$ to the corresponding vertices u_1, u_2. Furthermore, assign the labels $\frac{-3n-5}{2}, \frac{-3n-7}{2}, \ldots, -2n$ and $\frac{3n+5}{2}, \frac{3n+7}{2}, \ldots, 2n$ to the corresponding vertices $u_3, u_5, \ldots, u_{n-2}$ and $u_4, u_6, \ldots, u_{n-1}$. Fix the label 1 to u_n.
Case (ii) : For even n

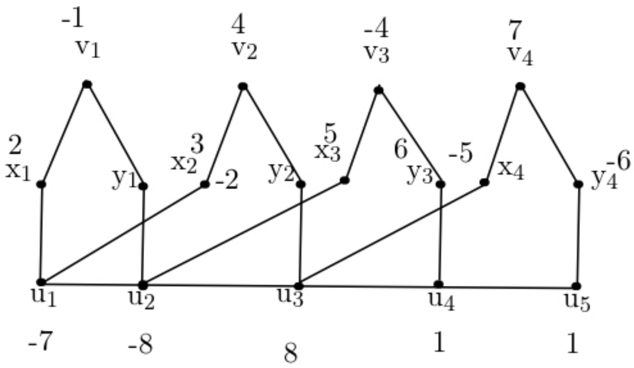

Figure 13: PMC-labeling of the irregular pentagonal snake graph IP_5.

Also, assign the labels $2, 5, \dots, \frac{3n-2}{2}$ and $-2, -5, \dots, \frac{-3n+2}{2}$ to the corresponding vertices $x_1, x_3, \dots, x_{n-1}$ and $x_2, x_4, \dots, x_n$. Next, assign the labels $-1, -4, \dots, \frac{-3n+4}{2}$ and $4, 7, \dots, \frac{3n+2}{2}$ to the corresponding vertices $y_1, y_3, \dots, y_{n-1}$ and $y_2, y_4, \dots, y_n$. Fix the label $\frac{-3n}{2}$ with v_1. Assign the labels $3, 6, \dots, \frac{3n}{2}$ and $-3, -6, \dots, \frac{-3n+6}{2}$ to the corresponding vertices $v_2, v_4, \dots, v_n$ and $v_3, v_5, \dots, v_{n-1}$. Further, assign the labels $\frac{3n+4}{2}, \frac{-3n-2}{2}$ to the corresponding vertices u_1, u_3. Assign the labels $\frac{-3n-4}{2}, \frac{-3n-6}{2}, \dots, -2n$ and $\frac{3n+6}{2}, \frac{3n+8}{2}, \dots, 2n$ to the corresponding vertices $u_2, u_4, \dots, u_{n-2}$ and $u_5, u_7, \dots, u_{n-1}$. Fix the label 1 to u_n. In both cases, $\bar{S}_{\lambda_1} = 3n = \bar{S}_{\lambda_1^c}$.

□

Example 3.13. Figure 14 illustrates the PMC-labeling of the rectangular winged prism graph $RWPR_5$.

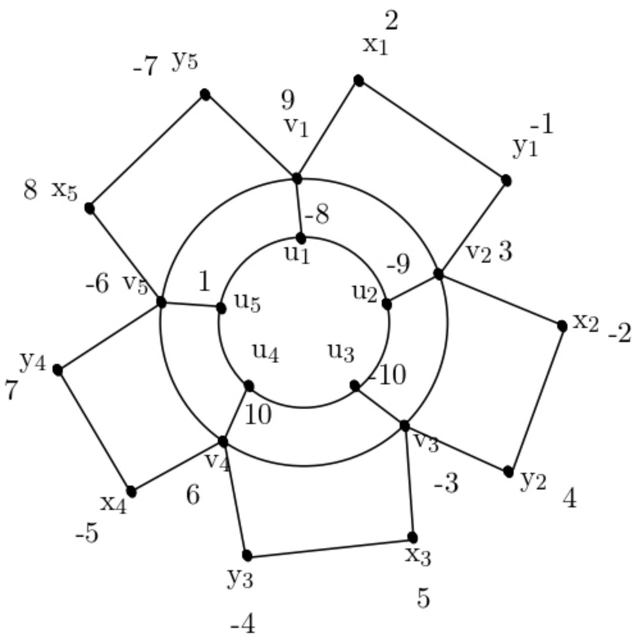

Figure 14: PMC-labeling of the rectangular winged prism graph $RWPR_5$.

Theorem 3.14. The W−graph WB_n is a PMC-graph only for all values of $n \leqslant 5$.

Proof. Denote by $V(WB_n) = \{u_0, v_0, u_i, v_j \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n-1\}$ and $E(WB_n) = \{u_0u_i, v_0v_j, u_nv_0 \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n-1\}$ respectively, the vertex and edge sets of the $W-$graph WB_n. Clearly, WB_n has $2n+1$ vertices and $2n$ edges.
Case (i) : For $n = 2$
Now, assign the labels $-1, 1, 2$ and $-1, -2$ to the corresponding vertices u_0, u_1, u_2 and v_0, v_1. Then, $\bar{S}_{\lambda_1} = 2 = \bar{S}_{\lambda_1^c}$.
Case (ii) : For $n = 3$
So assign the labels $-1, 1, 2, 3$ and $-2, -3, 1$ to the corresponding vertices u_0, u_1, u_2, u_3 and v_0, v_1, v_2. Then, $\bar{S}_{\lambda_1} = 3 = \bar{S}_{\lambda_1^c}$.
Case (iii) : For $n = 4$
Next, assign the labels $-1, 1, 2, 3, 4$ and $-2, -3, -4, 4$ to the corresponding vertices u_0, u_1, u_2, u_3, u_4 and v_0, v_1, v_2, v_3. Then, $\bar{S}_{\lambda_1} = 4 = \bar{S}_{\lambda_1^c}$.
Case (iv) : For $n = 5$
Assign the labels $-1, 1, 2, 3, -2, 4$ and $-3, -4, -5, 4, 5$ to the corresponding vertices $u_0, u_1, u_2, u_3, u_4, u_5$ and v_0, v_1, v_2, v_3, v_4. Then, $\bar{S}_{\lambda_1} = 5 = \bar{S}_{\lambda_1^c}$.
Case (v) : For $n \geqslant 6$
If possible, let the $W-$graph WB_n be a PMC-graph. The possible outcomes are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$ if the edge uv receives the label one. Therefore, $\bar{S}_{\lambda_1} \leqslant 5$. Consequently, $\bar{S}_{\lambda_1^c} \geqslant 2n - 5$. Thus, $\bar{S}_{\lambda_1^c} - \bar{S}_{\lambda_1} \geqslant 2n - 5 - 5 = 2n - 10 \geqslant 2 > 1$, a contradiction. □

Example 3.15. Figure 15 illustrates the PMC-labeling of the W-graph WB_4.

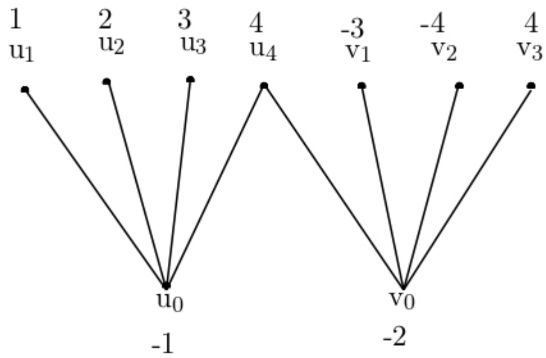

Figure 15: PMC-labeling of the W-graph WB_4.

Theorem 3.16. The triangular winged prism graph $TWPR_n$ is a PMC-graph only for all values of $n \geqslant 5$.

Proof. Denote by $V(TWPR_n) = \{u_i, v_i, w_i \mid 1 \leqslant i \leqslant n\}$ and $E(TWPR_n) = \{u_iv_i, v_iw_i \mid 1 \leqslant i \leqslant n\} \cup \{u_iu_{i+1}, u_nu_1, v_iv_{i+1}, v_nv_1, w_iv_{i+1}, w_nv_1 \mid 1 \leqslant i \leqslant n-1\}$ respectively, the vertex and edge sets of the triangular winged prism graph $TWPR_n$. Eventually, $TWPR_n$ has $3n$ vertices and $5n$ edges. We consider three cases:
If possible, let λ be a PMC-graph. The possible outcomes are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$ if the edge uv receives the label one.
Case (i) : For $n = 3$
Here, 6 is the maximum number of edges that can be labeled as 1. Then $\bar{S}_{\lambda_1} \leqslant 6$. Consequently, $\bar{S}_{\lambda_1^c} \geqslant 8$. Therefore $\bar{S}_{\lambda_1^c} - \bar{S}_{\lambda_1} \geqslant 8 - 6 = 2 > 1$, a contradiction.
Case (ii) : For $n = 4$
Now, 9 is the maximum number of edges that can be labeled as 1. That's $\bar{S}_{\lambda_1} \leqslant 9$. Consequently, $\bar{S}_{\lambda_1^c} \geqslant 11$. Therefore $\bar{S}_{\lambda_1^c} - \bar{S}_{\lambda_1} \geqslant 5n - 4 - (2n + 4) = 11 - 9 \geqslant 2 > 1$, a contradiction.
Case (iii) : For $n \geqslant 5$

Let $\lambda(v_1) = -n$. So assign the labels $-1, -2, \ldots, -n+1$ and $2, 3, \ldots, n+1$ to the corresponding vertices $v_2, v_3, \ldots, v_n$ and $w_1, w_2, \ldots, w_n$. We consider four cases:
Subcase $(i) : n \equiv 0 \pmod 4$
Now, assign the labels $n+2, n+3, \ldots, \frac{5n+8}{4}$ and $-n-1, -n-2, \ldots, \frac{-5n}{4}$ to the corresponding vertices $u_1, u_3, \ldots, u_{\frac{n+2}{2}}$ and $u_2, u_4, \ldots, u_{\frac{n}{2}}$. Fix the label $\frac{-5n-4}{4}$ with $u_{\frac{n+6}{2}}$. Assign the labels $\frac{-5n-8}{4}, \frac{-5n-12}{4}, \ldots, \frac{-3n}{2}$ and $\frac{5n+12}{4}, \frac{5n+16}{4}, \ldots, \frac{3n}{2}$ to the corresponding vertices $u_{\frac{n+4}{2}}, u_{\frac{n+8}{2}}, \ldots, u_{n-2}$ and $u_{\frac{n+10}{2}}, u_{\frac{n+14}{2}}, \ldots, u_{n-1}$. Fix the label 1 with u_n.
Subcase $(ii) : n \equiv 1 \pmod 4$
Also, assign the labels $n+2, n+3, \ldots, \frac{5n+5}{4}$ and $-n-1, -n-2, \ldots, \frac{-5n-1}{4}$ to the corresponding vertices $u_1, u_3, \ldots, u_{\frac{n-1}{2}}$ and $u_2, u_4, \ldots, u_{\frac{n+1}{2}}$. Fix the label $\frac{-5n-5}{4}$ with $u_{\frac{n+3}{2}}$. Furthermore, assign the labels $\frac{-5n-9}{4}, \frac{-5n-13}{4}, \ldots, \frac{-3n+1}{2}$ and $\frac{5n+9}{4}, \frac{5n+13}{4}, \ldots, \frac{3n-1}{2}$ to the corresponding vertices $u_{\frac{n+5}{2}}, u_{\frac{n+9}{2}}, \ldots, u_{n-3}$ and $u_{\frac{n+7}{2}}, u_{\frac{n+11}{2}}, \ldots, u_{n-2}$. Assign the labels $1, \frac{-3n+1}{2}$ to the vertices u_{n-1}, u_n respectively. If $n = 5$, $\lambda(u_n) = 1$.
Subcase $(iii) : n \equiv 2 \pmod 4$
Then, assign the labels $n+2, n+3, \ldots, \frac{5n+6}{4}$ and $-n-1, -n-2, \ldots, \frac{-5n-2}{4}$ to the corresponding vertices $u_1, u_3, \ldots, u_{\frac{n}{2}}$ and $u_2, u_4, \ldots, u_{\frac{n+4}{2}}$. Fix the label $\frac{-5n-6}{4}$ with $u_{\frac{n+6}{2}}$. So assign the labels $\frac{-5n-10}{4}, \frac{-5n-14}{4}, \ldots, \frac{-3n}{2}$ and $\frac{5n+10}{4}, \frac{5n+14}{4}, \ldots, \frac{3n}{2}$ to the corresponding vertices $u_{\frac{n+8}{2}}, u_{\frac{n+12}{2}}, \ldots, u_{n-2}$ and $u_{\frac{n+10}{2}}, u_{\frac{n+14}{2}}, \ldots, u_{n-1}$. Fix the label 1 with u_n.
Subcase $(iv) : n \equiv 3 \pmod 4$
Further, assign the labels $n+2, n+3, \ldots, \frac{5n+7}{4}$ and $-n-1, -n-2, \ldots, \frac{-5n+1}{4}$ to the corresponding vertices $u_1, u_3, \ldots, u_{\frac{n+1}{2}}$ and $u_2, u_4, \ldots, u_{\frac{n-1}{2}}$. Fix the label $\frac{-5n-3}{4}$ with $u_{\frac{n+5}{2}}$. More over, assign the labels $\frac{-5n-7}{4}, \frac{-5n-11}{4}, \ldots, \frac{-3n+1}{2}$ and $\frac{5n+11}{4}, \frac{5n+15}{4}, \ldots, \frac{3n-1}{2}$ to the corresponding vertices $u_{\frac{n+3}{2}}, u_{\frac{n+7}{2}}, \ldots, u_{n-3}$ and $u_{\frac{n+9}{2}}, u_{\frac{n+13}{2}}, \ldots, u_{n-2}$. Assign the labels $1, \frac{-3n+1}{2}$ to the vertices u_{n-1}, u_n respectively. □

Table 2: The following table 1 demonstrates the vertex labelling λ is a PMC-labeling of the triangular winged prism graph $TWPR_n$, for $n \geqslant 5$.

n	$\bar{S}_{\lambda_1}$	$\bar{S}_{\lambda_1^c}$
$n \equiv 0 \pmod 4$	$\frac{5n}{2}$	$\frac{5n}{2}$
$n \equiv 1 \pmod 4$	$\frac{5n-1}{2}$	$\frac{5n+1}{2}$
$n \equiv 2 \pmod 4$	$\frac{5n}{2}$	$\frac{5n}{2}$
$n \equiv 3 \pmod 4$	$\frac{5n-1}{2}$	$\frac{5n+1}{2}$

Example 3.17. Figure 16 illustrates the PMC-labeling of the triangular winged prism graph $TWPR_6$.

4. Conclusion

Labeled graphs are currently a popular research area for many scholars for their wide range of applications. In this paper, we have studied the pair mean cordial labeling behavior of some special graphs like the total graph of path, cycle, star, crown and comb and also we have examined the pair mean cordial labeling behavior of the triangular winged prism graph, rectangular winged prism graph, $W-$ graph and irregular pentagonal snake. It is an open area of research to derive similar results on different types of graph families.

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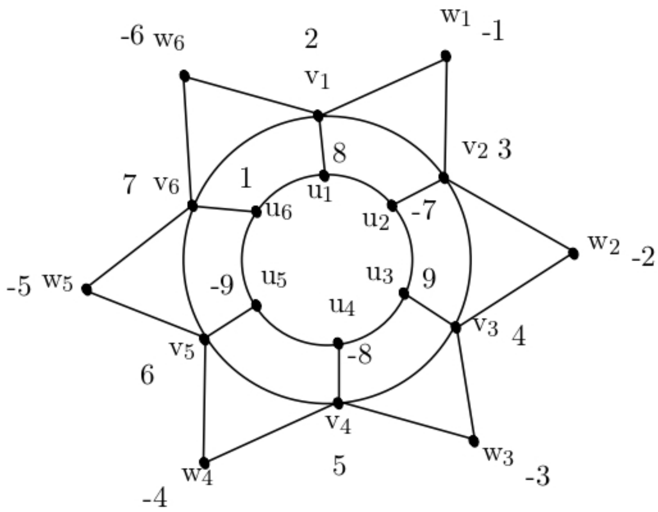

Figure 16: PMC-labeling of the triangular winged prism graph $TWPR_6$.

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